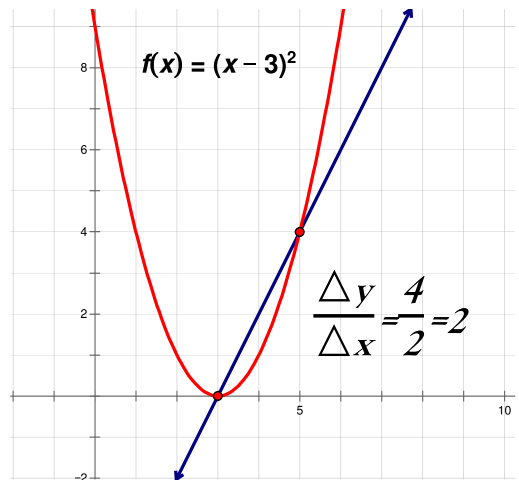
1E: Difference Quotient

Notes

A very important question in mathematics is how to find the slope of a line that is tangent to a curve. A tangent line touches a curve in exactly one point without crossing the curve. In formal Geometry courses we study lines that are tangent to a circle and discover some interesting results such as the fact that a tangent line to a circle is perpendicular to the radius that intersects it.

In the previous lesson, we found the average rate of change. Graphically, the rate of change on the interval $[a, b]$ represents the slope of the secant line through the points a and b (as in the example to the right)



We now want to choose two points that are closer and closer together. The example to the right shows a secant line for the interval [3,5]. We may want to also consider the intervals

$$[3,4], [3, 3.5], [3,3.1], \dots, [3,3.000001].$$

What would happen to the slope of the secant line of

$$f(x) = (x - 3)^2$$

over these intervals listed above? (Use the graph on the right to help.)

To generalize this, we will set the distance between the two endpoints and call this h. This leaves us with the interval $[x, x + h]$. Now, we will find is the rate of change for $f(x)$ on the interval $[x, x + h]$. We will call this function $D(x)$ the ***Difference Quotient***.

$$D(x) = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x}$$

or

Difference Quotient: $$D(x) = \frac{f(x+h) - f(x)}{h}$$

Example:
Find and simplify the difference quotient function D(x) for each function.

a) $f(x) = 2x + 3$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{(2(x+h)+3) - (2x+3)}{h} = \frac{2x + 2h + 3 - 2x - 3}{h}$$
$$= \frac{2h}{h} = 2$$

b) $g(x) = x^2$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h}$$
$$= 2x + h$$

c) Use the result of (b) to find the rate of change for $g(x) = x^2$ on [1,3].

For the interval $[x, x+h] = [1, 3]$, we see $x = 1$ and $h = 3 - 1 = 2$ which is the length of the interval. So for this $x = 1$, we have

$$D(x) = 2x + h$$
$$D(1) = 2(1) + (2) = 4$$

This is the slope of the secant line for $g(x) = x^2$ on the interval $[1, 3]$ is 4.

Limit as $h \to 0$

In the difference quotient function, h cannot be 0 since it is in the denominator. However, an important question to ask is

"What does the value of the function become as h goes to 0?"

Example:
Consider the function $D(x)$ above for $g(x) = x^2$.
What does this function become as h goes to zero?

$$\lim_{h \to 0}(2x + h) = 2x$$

This means that the slope of the line tangent to the graph of $g(x) = x^2$ at some value of x is equal to 2x.

****Key Point:** The value of $D(a)$ as $h \to 0$ is the ____Slope____ of the line *tangent* to the curve at $(a, f(a))$.**

Assignment: SOLUTIONS

For each of the functions below, find and simplify D(x).

1. $a(x) = 3x - 2$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{(3(x+h) - 2) - (3x - 2)}{h} = \frac{3x + 3h - 2 - 3x + 2}{h}$$
$$= \frac{3h}{h} = 3$$

2. $b(x) = \frac{1}{2}x + 4$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{\left(\frac{1}{2}(x+h) + 4\right) - \left(\frac{1}{2}x + 4\right)}{h} = \frac{\frac{1}{2}x + \frac{1}{2}h + 4 - \frac{1}{2}x - 4}{h}$$
$$= \frac{\frac{1}{2}h}{h} = \frac{1}{2}$$

3. $c(x) = mx + b$. (treat m and b as numbers)

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{(m(x+h) + b) - (mx + b)}{h}$$
$$= \frac{mx + mh + b - mx - b}{h} = \frac{mh}{h} = m$$

4. $h(x) = 2x^2$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{(2(x+h)^2) - (2x^2)}{h} = \frac{2(x^2 + 2xh + h^2) - 2x^2}{h}$$
$$= \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h} = \frac{4xh + 2h^2}{h} = 4x + 2h$$

5. $j(x) = 3x^2$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{(3(x+h)^2) - (3x^2)}{h} = \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$
$$= \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} = \frac{6xh + 3h^2}{h} = 6x + 3h$$

6. $k(x) = x^2 + x$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{((x+h)^2 + (x+h)) - (x^2 + x)}{h}$$
$$= \frac{(x^2 + 2xh + h^2 + x + h) - (x^2 + x)}{h} = \frac{2xh + h^2 + h}{h}$$
$$= 2x + h + 1$$

7. $m(x) = x^2 + 5$

$$D(x) = \frac{f(x+h) - f(x)}{h} = \frac{((x+h)^2 + 5) - (x^2 + 5)}{h} = \frac{(x^2 + 2xh + h^2 + 5) - (x^2 + 5)}{h}$$
$$= \frac{2xh + h^2}{h} = 2x + h$$