4B-2: Using Exponential and Logistic Functions

The general form for an exponential function is

$$f(x) = a \cdot b^x$$

Where a is the initial value and the base b determines the growth or decay. In this lesson we will investigate some applications of exponential functions and a related function called the logistic function.

Modeling with Exponential Functions

Our first important application of exponential functions is *population growth and decay*.

Definition: The **Growth Factor** is defined as $1 + r$, where r is the percentage rate of change per unit of time. If r is negative, then $1 + r$ is called the **Decay Factor**.

Population Growth Model

A population P can be modeled over time t by the function

$$P(t) = P_0(1 + r)^t$$

where r is the percentage rate of change per unit of time and initial population P_0.

Explore

Suppose we have an amoeba population of 10 specimens that reproduces by doubling every day. That is, it has a rate of change of 100% per day.

a) Make a table for the first 5 days.

Day	0	1	2	3	4	5
Population	10					

b) Write an exponential equation to model the population P as a function of time t.

c) Use your equation to find the population after 10, 20, and 30 days

d) Write an equation that we can use to find the day when the population will reach 1 million. Use logarithms to solve this equation.

Example

Suppose we have an exponential function ($f(x) = a \cdot b^t$) that passes through the point (0,10) and (5,2.5). Find the equation for the function.

Step 1: Determine the initial value from the given information:

Step 2: Substitute the initial value and the second point to solve for b.

Modeling with Logistic Functions

Populations like the one in the exploration above will often grow exponentially, but there comes a point where growth will slow down due to limiting factors like resources or space. To better model true population growth, we can use a function called a *logistic growth function.*

Logistic Growth Function:
With $a, b, c,$ and k as positive constants and $b < 1$, we define logistic growth as

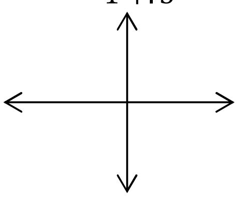
$$f(x) = \frac{c}{1 + a \cdot b^x}, \qquad \text{or} \qquad f(x) = \frac{c}{1 + a \cdot e^{-kx}}$$

where the constant c is the *limit of growth*.

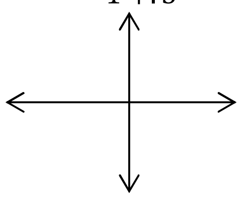
Explore Logistic graphs.

Graph the following functions in your calculator and sketch the graph.

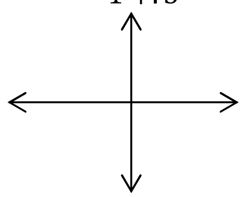
$y = \frac{1}{1 + .5^x}$



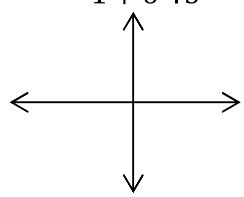
$y = \frac{3}{1 + .5^x}$



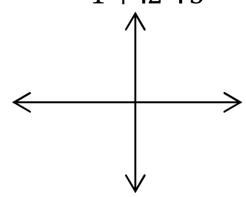
$y = \frac{10}{1 + .5^x}$



$y = \frac{10}{1 + 6 \cdot .5^x}$



$y = \frac{10}{1 + .2 \cdot .5^x}$



Example:

Find the limit of growth and the y-intercept for each of the functions above.

a) $y = \frac{1}{1+.5^x}$ *limit of growth =* *y-intercept=*

b) $y = \frac{3}{1+.5^x}$ *limit of growth =* *y-intercept=*

c) $y = \frac{10}{1+.5^x}$ *limit of growth =* *y-intercept=*

d) $y = \frac{10}{1+6 \cdot .5^x}$ *limit of growth =* *y-intercept=*

e) $y = \frac{10}{1+.2 \cdot .5^x}$ *limit of growth =* *y-intercept=*

Exercises

For Exercises 1 and 2, tell whether the function is an exponential growth function or an exponential decay function.

1. $f(t) = 35 \cdot 4^t$
2. $g(t) = 45 \cdot .95^t$

3. Write an exponential function with an initial population of 52 and increases 2.3% each year. When will the population be greater than 100?

Write the exponential function of the form $f(x) = a(b^x)$ that passes through the given points.

4. (0,4) and (3,665.5)

5. $(0, -3)$ and $(5, -50421)$

6. A population of bacteria has exponential growth modeled by $g(t) = 20e^{.25t}$ for time t in days.
   a. What is the initial population?

   b. When will the population reach 1000?

7. The half-life of Plutonium-238 used in nuclear generators is 88 years. The exponential function to model the amount of plutonium-238 $M(t) = 50(0.5)^{(t/88)}$ in a certain generator after t years.
   a. What was the initial mass (in grams) of Plutonium-238 in the generator?

   b. When will the amount of Plutonium-238 be less than 10 grams

For the following logistic functions in 8 and 9, find the limit to its growth and find the y intercept

8. $f(x) = \frac{70}{1+3(0.2)^x}$

9. $f(x) = \frac{200}{1+5(4)^x}$

10. Find the logistic function with initial value=10, limit of growth=40, and passes through (1,20),