Pre-Calculus Name: Date: Period:

3C: Polynomial Equations & The Fundamental Theorem of Algebra

We have seen that a polynomial function of degree n can have at most n real zeros where the graph intersects the x-axis. However, not every function has n real zeros, and some functions (like $y = x^2 + 2$) have no zeros!

So, does this mean that an equation of the form $0 = 2x^2 + 8$ does not have a real solution because the graph of $f(x) = 2x^2 + 8$ does not have any x-intercepts? We know from our experience that it actually has two solutions, but they are both *complex* solutions. This is a result of the ***Fundamental Theorem of Algebra***.

Fundamental Theorem of Algebra:

A polynomial function of degree $n > 0$ has exactly n complex zeros.
(Some of these may be repeated zeros.)

Try This: Find all the zeros of $f(x) = 2x^2 + 8$.

Linear Factorization Theorem

Every polynomial function of degree $n > 0$ can be factored into the form

$$f(x) = a(x - z_1)(x - z_2)\cdots(x - z_n)$$

where $z_1, z_2, \ldots, z_n$ are the complex zeros of $f(x)$. Keep in mind that if there are repeated zeros, some z_i values may be the same.

Factor it: Write the linear factorization of $f(x) = 2x^2 + 8$.

****Important Note: Complex zeros always come in conjugate pairs!**

i.e. **If $a + b$ i is a zero of a polynomial, then $a - b$ i is also a zero.**

Simplifying for Standard Form:

Write the polynomial in standard form, and identify the zeros of the function and the x-intercepts of the graph. Verify your answer with your calculator.

$$f(x) = (x + 1)(x + \sqrt{2}i)(x - \sqrt{2}i)$$

Key Connections:

The following statements about a polynomial function are all equivalent for any complex number k:

1. $(x - k)$ is a factor of $f(x)$.
2. k is a zero of the function f.
3. $f(k) = 0$, which implies that $x = k$ is a root or x-intercept of the graph of $f(x)$.

Finding All Zeros (including complex ones!)

We have seen that real zeros are x-intercepts, so we can find these real zeros using a graphing calculator.

Consider this:

a) Use your graphing calculator to find the real zeros of $y = 2\,x^4 - 4\,x^3 - x^2 + 3\,x - 2$

b) Does $g(x)$ have any complex zeros with imaginary parts? Explain.

To find all n zeros of a polynomial function $f(x) = a_1 x^n + a_2 x^{n-1} + \cdots + a_n$, we need to do the following:

Finding all complex zeros of a polynomial:

1. Use the rational root theorem or a graphing calculator to find a possible real zero z_1.
2. Use synthetic division to divide the polynomial by z_1.
   If we get a remainder of 0, then z is a zero of the function and this gives us a factorization
   $$f(x) = (x - z_1)(q_1 x^{n-1} + q_2 x^{n-2} + \cdots q_{n-1})$$
   where $q_1, q_2, \ldots, q_{n-1}$ are the coefficients of the quotient that result from the synthetic division.
3. If the quotient is a quadratic, use the techniques we know to solve $0 = q_1 x^2 + q_2 x + q_3$. If you can factor the quotient, do this.
4. If the quotient in (2) is not a quadratic and we cannot factor it other ways, then repeat steps (1) and (2) until one of these is true.

Try it out.

a) Find all the zeros of this functions.

$$f(x) = 4x^3 + 8x^2 + 7x + 14$$

b) Solve the equation $-3x^4 - 2x^3 = -35x^2 + 14x - 24$.
(Note: this really is the same as (a). Try making the left side 0 first.)

c) Solve the equation $x^4 + 10x^3 = -26x^2 - 10x - 25$

d) How many complex solutions should the equation in (c) have according to the Fundamental Theorem of Algebra?

Did you find *all* the solutions to this equation? Explain?