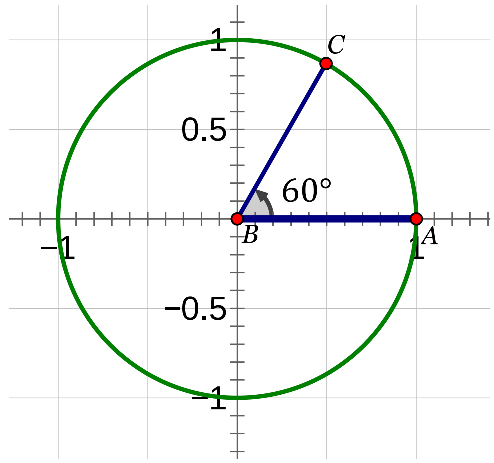
7A: Angles and Arc Length

Circular Motion

Many patterns in nature follow reoccurring sequences such as the cycle of your blood as your heart pumps, the path of the earth around the sun, the swinging of a pendulum, and the rise and fall of the ocean's tides. To model these, we use angles that revolve around a central point, namely the origin of the coordinate plane.

A **central angle** is one with its vertex at the center of a circle. When we measure angles in the coordinate plane, we will place angles in ***standard position*** with one side, the ***initial side***, on the positive x-axis. The other side of the angle is called the ***terminal side***. The measure of the angle is the amount of rotation between the initial side and the terminal side.



Traditionally circles have been divided into 360 parts or **degrees** (°)since some ancient calendars had 360 days. Every degree is divided into 60 **minutes** (′). Every minute is divided into 60 **seconds** (″). This type of measurement is called Degree-Minutes-Seconds form or DMS form.

Example

a) Convert $52°34'25"$ to degrees.

b) Convert $37.425°$ to DMS form

When measuring angles in standard position, we can rotate clockwise or counter-clockwise.

Positive angles are measured going *counter-clockwise*.

Negative angles are measured going *clockwise.*

This means that we can have two angles that have the same initial and terminal sides, but different measures, these are called ***coterminal sides***.

Try It: Find two ways (one positive and one negative) to name the angle in the figure above.

Example: Find the angle measures of two coterminal angles for each of the following:

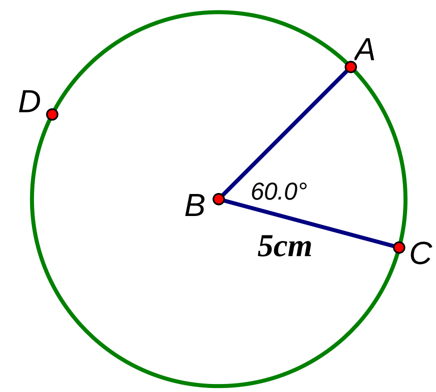
a) $30°$

b) $-70°$

c) $90°$

Arc Length - Degrees

The measure of an arc is determined by the measure of its intercepted arc. When working with angles and arcs it is useful to be able to find the actual curved length of an arc.



Consider this. Answer the following questions using the circle to the right.

a) Find the circumference of circle B.

b) What is the measure (in degrees) of $\widehat{AC}$?

c) What fraction of the circumference is $\widehat{AC}$?

d) Find the length (in cm) of $\widehat{AC}$.

e) Find the length of $\widehat{ADC}$.

Arc Length Formula (in degrees):
If θ is a central angle **in degrees** of radius r, the length s of the intercepted arc is given by

$$s = \frac{\theta}{360} \cdot 2\pi r \qquad or \qquad s = \frac{\pi r \theta}{180}$$

Try These

a) Find the length of an arc of a 3 cm circle made by a 120° angle.

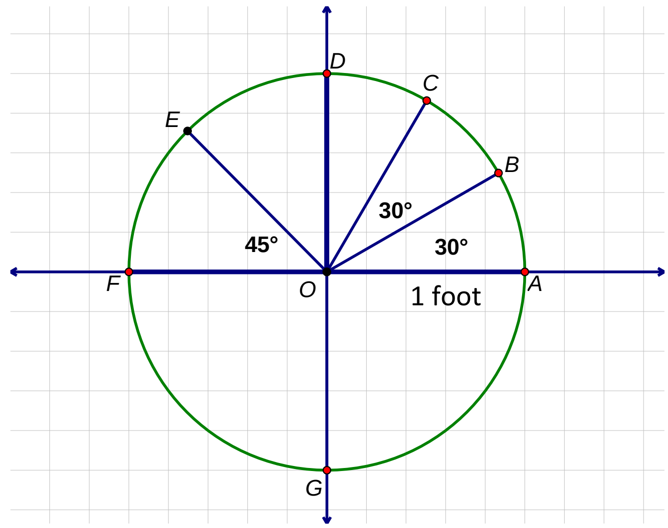
b) If an arc has a length of 5 in. and an angle of 55°, find the radius of the circle.

c) A 10 in. radius gear needs to turn 8 in. How much of an angle does the gear need to turn?

d) Two boats leave a harbor. Boat A leaves at a heading of 40° east of north, and boat B leaves at a heading 50° east of north. After they travel 2 miles, how far apart are they (in arc length)?

Another way to measure angles

In the summer of 1985, Ian the flea has joined the Fabulous Flying Flea Circus and his act is to walk the circular tightrope called the "Ring of Fire" (because it was painted red). So, the ring master set up the circular tightrope with a radius of
1 foot (the largest tightrope ever attempted by any flea).



If Ian travels all the way around the circular ring, traveling in a full 360 ° arc, exactly how far will he walk (answer in terms of π)?

The crowd was amazed every time Ian performed this death-defying feat, at he became very famous. Onlookers watched in amazement and said, "That's rad!" So, he became known as "Rad-Ian", and the impossible tightrope stunt he performed walking around the 1 foot circle became known as the "2 π Rad-Ian". This was soon just shortened to the *2π Radian* stunt.

All the other tightrope fleas wanted to be as good as Rad-Ian, but most could not make it all the way around the circle. Every stunt was now measured by the Radian. So, when Semi-Circle Sally went half way around the circle, she travelled 180° around the circle, or 1π Radians.

Each of the arcs bellow is the path that will be travelled by a circus flea. Determine the arc measure in degrees, the length of the arc, and the measure of the arc in "Radians".

Try This
Using the circle to the right, name the arc intercepted by the angle, find the arc measure in degrees, and find the arc length.

Central Angle	Intercepted Arc Name	Arc measure in degrees	Arc Length
$\angle AOB$			
$\angle AOC$			
$\angle AOD$			
$\angle AOE$			
$\angle AOF$			
$Reflex\ angle\ \angle AOG$			

1 Radian - the measure of an angle that intercepts an arc the same length as its radius.

Radian Measure

Definition 1: If an angle intercepts an arc of length s in a circle of radius r, then

$$radian\ measure = \frac{s}{r}$$

Definition 2: *The radian measure of an angle is the length of the intercepted arc of the unit circle (radius =1)*

Converting Angle Measures: $\mathbf{180° = 1\pi\ radians}$

To convert radians to degrees, multiply by $\frac{180°}{\pi\ radians}$

To convert degrees to radians, multiply by $\frac{\pi\ radians}{180°}$

Try These

a) Convert $160°$ to radians.

b) Convert $75°35'$ to radians.

c) Convert $\frac{7\pi}{6}$ to degrees.

d) Convert $\frac{3\pi}{10}$ to degrees.

Arc Length - Radians

As we just saw, the radian measure of an angle is determined by the length of the intercepted arc in a unit circle. Since radian measure is already an arc length, we can easily find the arc length of any arc intercepted by a central angle that is measured in radians by a simple scale factor.

Arc Length Formula (in radians):
If θ is a central angle **in radians** of radius r, the length s of the intercepted arc is given by

$$s = r\theta$$

We see that radians is a much more direct and natural way to find arc length.

Try These

a) Find the length of an arc of a 3 cm radius circle made by a π radian angle.

b) Find the length of an arc of a 3 cm circle made by a $\frac{2\pi}{3}$ radian angle.

c) Find the radius needed to make a 5 in. arc at a $\frac{2\pi}{5}$ radian angle.

Angular velocity: *Speed measured in rotations (or revolutions) per unit of time. As a point moves along a circle of radius* r, *the angular velocity,* ω, *can be found as the angular rotation,* θ, *per unit of time,* t.

$$\omega = \frac{\theta}{t}$$

Linear velocity: *Speed measured in length per unit of time. The linear velocity,* v, *of a point can be found as the distance traveled, arc length* s, *per unit of time,* t.

$$v = \frac{s}{t}$$

Example:

a) *A trucks wheels have a diameter of 36 inches. If the wheels are rotating at 500 rpm, find the speed of the car in miles per hour. Remember,* $1\ revolution = 2\pi\ radians$.

b) *Find the angular velocity of a rock that is stuck in the trucks tire in radians per second.*