Pre-Calculus Name: Date: Period:

5B: Solving 3-Variable Linear Systems

$X+Y=10 \rightarrow$ line

$X+Y+Z=10 \rightarrow$ plane

From Two Variables to Three

We have now seen that if we have two equations with the same two variables x and y, we can consider this a system of two equations and solve for the ordered pair (x, y) that is a solution of both equations. (Note that we must have two equations to solve for a unique solution.)

Geometrically, we found that the solution is the point of intersection of the two lines. This also shows that we can have 0, 1, or infinite solutions depending on where (or if) they intersect.

Example Solve this system by **substitution**

$$\begin{aligned} 2x - y + z &= 0, & E_1 \\ x + 2y + 4z &= 4, & E_2 \\ z &= x - y & E_3 \end{aligned}$$

Step 1: Substitute E_3 into E_1 and E_2 to write two new equations in terms of x and y.

E_1: $2x-y+(x-y)=0$

$3x-2y=0$

E_2: $x+2y+4(x-y)=4$

$x+2y+4x-4y=4$

$5x-2y=4$

Step 2: Solve these two equations for x and y.

$$\begin{array}{r} 3x-2y=0 \\ -\quad 5x-2y=4 \\ \hline -2x=-4 \\ x=2 \end{array} \qquad \begin{array}{r} 15x-10y=0 \\ -\quad 15x-6y=12 \\ \hline -4y=-12 \\ y=3 \end{array}$$

Step 3: Use these values to solve for z.

$z=x-y$

$z=(2-3)=-1$

$\boxed{\begin{array}{l} x=2 \\ y=3 \\ z=-1 \end{array}}$ $\boxed{(2,3,-1)}$

Example Solve this system using Gaussian Elimination

E_1 $2x - y + z = 0$
E_2 $x + 2y + 4z = 4$
E_3 $-x + y + z = 0$

$-x + 3 + (-1) = 0$
$\boxed{x = -2}$

$E_2 + E_3$: $3y + 5z = 4$

$+$ E_1: $2x - y + z = 0$
$2E_3$: $-2x + 2y + 2z = 0$

$y + 3z = 0$

$3y + 5z = 4 \rightarrow 3y + 5z = 4$
$y + 3z = 0 \xrightarrow{\cdot -3} -3y - 9z = 0$ $+$

$-4z = 4$
$\boxed{z = -1}$

$y + 3(-1) = 0$
$y + -3 = 0$
$\boxed{y = +3}$

Example Solve this system using matrices and elementary row operations.

$2x - y + z = 0$
$x + 2y + 4z = 4$
$-x + y + z = 0$

$R_2 + R_3$ $\quad R_1 - 2R_2$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 2 & 4 & 4 \\ -1 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 2 & 4 & 4 \\ 0 & 3 & 5 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & -5 & -7 & -8 \\ 0 & 3 & 5 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & -15 & -21 & -24 \\ 0 & 15 & 25 & 20 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & -5 & -7 & -8 \\ 0 & 0 & 4 & -4 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & -5 & -7 & -8 \\ 0 & 0 & 1 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & 1 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] = \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]}$$

Example Solve this system using matrices and elementary row operations.
On Calculator

$x + y + z = 10$
$2x - 3y + z = 5$
$x + y + z = 10$ $\quad \cdot 2$: $2 \quad 2 \quad 2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & -3 & 1 & 5 \\ 2 & 2 & 2 & 20 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & .8 & 7 \\ 0 & 1 & .2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Dependent System

$x + .8z = 7$
$y + .2z = 3$
$z = z$

Exercises

1. Solve this system by substitution

$$3x - y + 2z = -2$$
$$y + 3z = 3$$
$$2z = 4$$

2. Solve using Gaussian Elimination.

$$x - y + z = 0$$
$$2x - 3z = -1$$
$$-x - y + 2z = -1$$

$E_1 + E_3: -2y + 3z = -1$

$-2 \cdot E_1: -2x + 2y - 2z = 0$

$E_2: 2x - 3z = -1$

$2y - 5z = -1$

$E_1 - E_3: 2x - z = 1$

$E_2: 2x - 3z = -1$

$-2y + 3z = -1$

$2y - 5z = -1$

$-2z = -2$

$\boxed{z = 1}$

3. Solve the system using matrices and elementary row operations. (calculator)

$$2x + 3y - 2z = 2$$
$$3x + 3y - z = 6$$
$$-3y + 3z = 3$$

4. Solve the system using any method.

$$x + 2y = 4$$
$$3x + 4y = 5$$
$$2y + z = 8$$

$$x + 2y + 0z = 4$$
$$3x + 4y + 0z = 5$$
$$2y + z = 8$$