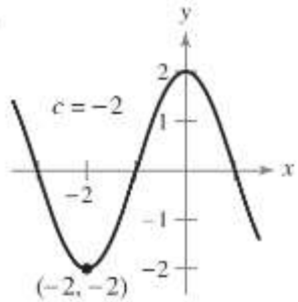
Calculus Name: Date:

1C Exercises

Continuity and the Intermediate Value Theorem

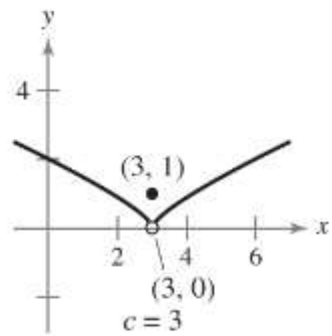
In Exercises 1–6, use the graph to determine the limit, and discuss the continuity of the function.

2.



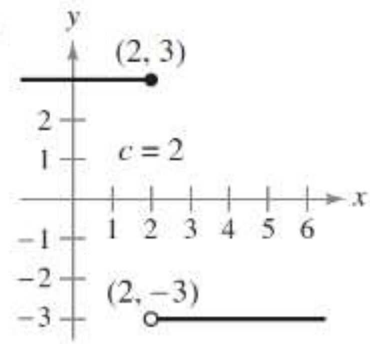
(a) $f(-2) = -2$
(b) $\lim_{x \to 3} f(x) = -2$
(c) $\lim_{x \to 3} f(x) = f(-2)$
The function is continuous at $x = -2$

3.



(a) $f(3) = 1$
(b) $\lim_{x \to 3} f(x) = 0$
(c) $\lim_{x \to 3} f(x) \neq f(3)$
The function is NOT continuous at $x = 3$

5.



(d) $f(2) = 3$
(e) $\lim_{x \to 2} f(x) = DNE$
(f) $\lim_{x \to 2} f(x) \neq f(2)$
The function is NOT continuous at $x = 2$, because the limit at 2 does not exist.

In Exercises 7–26, find the limit (if it exists). If it does not exist, explain why.

7. $\lim_{x \to 8^+} \frac{1}{x+8}$

$$\lim_{x \to 8^+} \frac{1}{x+8} = \frac{1}{8+8} = \frac{1}{16}$$

9. $\lim_{x \to 5^+} \frac{x-5}{x^2-25}$

$$\lim_{x \to 5^+} \frac{x-5}{x^2-25} = \lim_{x \to 5^+} \frac{1}{x+5} = \frac{1}{10}$$

17. $\lim_{x \to 3^-} f(x)$, where $f(x) = \begin{cases} \frac{x+2}{2}, & x \le 3 \\ \frac{12-2x}{3}, & x > 3 \end{cases}$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x+2}{2} = \frac{5}{2}$$

$$\lim_{\Delta x \to 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{x - (x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{-1}{x(x+\Delta x)}$$

$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

$$\lim_{\Delta x \to 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$$

18. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \geq 2 \end{cases}$

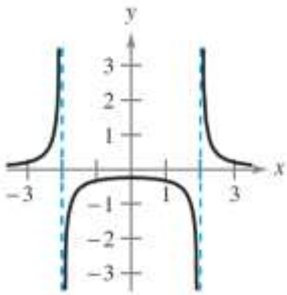
This problem will be checked on your assigment

24. $\lim_{x\to 2^+} (2x - [\![x]\!])$

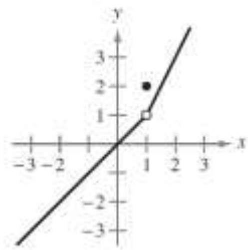
$\lim_{x\to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

In Exercises 27–30, discuss the continuity of each function.

27. $f(x) = \dfrac{1}{x^2 - 4}$



30. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$



This problem will be checked on your assigment

In Exercises 31–34, discuss the continuity of the function on the closed interval.

34. $g(x) = \dfrac{1}{x^2 - 4}$ $\quad [-1, 2]$

$$g(-1) = -\frac{1}{3}, \quad g(2) = undefined, \quad \lim_{x \to -1^+} g(x) = -\frac{1}{3}, \quad \lim_{x \to 2^-} g(x) = -\infty$$

Since the endpoint at 2 is undefined it is NOT continuous on the closed interval

Since the limit as $x \to -2$ is infinite, $g(x)$ is NOT continuous on the open interval either.

In Exercises 35– 60, find the -values (if any) at which is not continuous. Which of the discontinuities are removable?

41. $f(x) = 3x - \cos x$

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10}$

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

$$(a) \quad f(2) = -4$$

$$\lim_{x \to 2^-} f(x) = -4, \lim_{x \to 2^+} f(x) = -3$$

$$(b) \quad \lim_{x \to 2} f(x) = Does\ not\ exist$$

$$(c) \quad \lim_{x \to 2} f(x) \neq f(2)$$

Therfore, the $f(x)$ is NOT continuous at $x = 2$

63. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

67. $f(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

Writing In Exercises 83–86, explain why the function has a zero in the given interval.

83. $f(x) = \frac{1}{12}x^4 - x^3 + 4 \qquad [1, 2]$

86. $f(x) = -\frac{5}{x} + \tan\left(\frac{\pi x}{10}\right) \qquad [1, 4]$

$$f(1) = -5 + \tan\left(\frac{\pi}{10}\right) \approx -4.68$$

$$f(4) = -\frac{5}{4} + \tan\left(\frac{4x}{10}\right) \approx 1.83$$

By the Intermediate Value Theorem (IVT), since $f(1) < 0 < f(4)$, there must be a value c in $[1,4]$ such that $f(c) = 0$. i.e. a zero exists in $[1,4]$

In Exercises 91–94, verify that the Intermediate Value Theorem applies to the indicated interval and find the value of guaranteed by the theorem.

91. $f(x) = x^2 + x - 1, \quad [0, 5], \quad f(c) = 11$

94. $f(x) = \frac{x^2 + x}{x - 1}, \quad \left[\frac{5}{2}, 4\right], \quad f(c) = 6$

$$f\left(\frac{5}{2}\right) = \frac{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)}{\frac{5}{2} - 1} = \frac{\left(\frac{35}{4}\right)}{\frac{3}{2}} = \frac{35}{6}$$

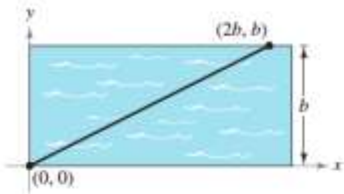
$$f(4) = \frac{4^2 + 4}{4 - 1} = \frac{20}{3}$$

Since $f\left(\frac{5}{2}\right) < 6 < f(4)$, by the IVT, there exists a c in $\left[\frac{5}{2}, 4\right]$ such that $f(c) = 6$

107. ***Déjà Vu*** At 8:00 A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at 8:00 A.M. he runs back down the mountain. It takes him 20 minutes to run up, but only 10 minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct. [*Hint:* Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.]

Saturday 8:00 A.M. Sunday 8:00 A.M.

114. ***Creating Models*** A swimmer crosses a pool of width b by swimming in a straight line from $(0, 0)$ to $(2b, b)$. (See figure.)



(a) Let f be a function defined as the y-coordinate of the point on the long side of the pool that is nearest the swimmer at any given time during the swimmer's crossing of the pool. Determine the function f and sketch its graph. Is f continuous? Explain.

(b) Let g be the minimum distance between the swimmer and the long sides of the pool. Determine the function g and sketch its graph. Is g continuous? Explain.