4E: Integration by Substitution

Antidifferentiation is a useful tool for figuring out simple integrals, but we need a multi-tool that is more versatile for integrating more general functions.

So, let's remember the chain rule..

$$\frac{d}{dx}\left[F\big(g(x)\big)\right] = F'\big(g(x)\big)g'(x)$$

Using the definition of antiderivative, and letting F be the antiderivative of f, we get

$$\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

So, the key is to let $u = g(x)$ and $du = g'(x)dx$

$$\int f(u)du = F(u) + C$$

The key is choosing the right
u substitution,
and this takes some practice.

$$\int f\big(g(x)\big)g'(x)\,dx$$

Inside Function

Derivative of Inside Function

Indefinite Integration with u – substitution

Try it: Use u-substitution to integrate. $\int (x^2+3)^3(2x)dx$

a) Define the u function

b) Find du

c) Substitute and integrate

d) Re-substitute for u

Sometimes after defining u, it helps to write x in terms of u.

Example Find $\int x\sqrt{2x-1}\,dx$

Change of Variables for Definite Integrals

When using u-substitution on a definite integral, we must also apply this substitution to the bounds.

Example Evaluate

$$\int_0^1 x(x^2+1)^3 dx$$

a) Define the u function

b) Find du

c) Substitute for integrand and bounds and integrate

A Couple Useful Theorems

Here is a useful theorem if the integrand is simply a product of a power of the function $g(x)$ and it's derivative $g'(x)$.

THEOREM 4.14 THE GENERAL POWER RULE FOR INTEGRATION

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example Find $\int \sin^5(x) \cos(x)\, dx$

THEOREM 4.16 INTEGRATION OF EVEN AND ODD FUNCTIONS

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$.
2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\, dx = 0$.

Remember:

- Even Functions: $f(-x) = f(x)$
- Odd Functions: $f(-x) = -f(x)$

Example Find the integrals

a) $\int_{-\pi}^{\pi} \sin x$

b) $\int_{-3}^{3} (x^4 + x^2) dx$