Calculus

Name: Date:

4A2: Summations of Areas

When finding the approximation of the area under a curve using rectangles, we found ourselves adding many areas together to make one large area. This process is called a ***summation.*** Very often, summations can have too many terms to write out, so we need a way to denote it.

Consider this: Find the sum of the first 100 positive integers. Describe your strategy for finding the solution.

$$1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100$$

In Sigma notation, we can write this like this

$$\sum_{i=1}^{100} i = 1 + 2 + \cdots + 100$$

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \dots, a_n$ can be written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index** of the summation, a_i is the ***i*th term** of the sum, and the **upper and lower bounds** are 1 and n.

Example Write out the terms of each summation

1. $\displaystyle\sum_{i=1}^{5} 2i =$

2. $\displaystyle\sum_{j=1}^{4} (2j + 1) =$

3. $\displaystyle\sum_{k=3}^{8} k^2$

4. $\displaystyle\sum_{i=1}^{n} \frac{1}{n}(n - 1) =$

5. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

Finding Formulas Use patterns to the following sums, write the general sum in sigma notation.

1. What the sum of the first 100 even numbers? How about the first n even numbers?

# of terms	1	2	3	4	5	…	100	…	n
Sum						…		…	

2. What is the sum of the first 100 natural numbers? How about the first n natural numbers?

# of terms	1	2	3	4	5	…	100	…	n
Sum						…		…	

3. What is the sum of the first 100 odd numbers? How about the first n odd numbers?

# of terms	1	2	3	4	5	…	100	…	n
Sum						…		…	

4. What is the sum of the first 100 square numbers? How about the first n square numbers?

# of terms	1	2	3	4	5	…	100	…	n
Sum	1	5	14	30	55	…		…	

Remember that Sigma notation is just a shortcut for writing out a long string of terms of a sum. So, we can do algebra with the Sigma notation and save a lot of writing.

Properties of Summations

Using the distributive and associative properties of addition, we can prove these important properties (k is a constant).

$$\sum_{i=1}^{n} k\,a_i = k\sum_{i=1}^{n} a_i \qquad and \qquad \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

We also have these useful theorems that are a result of what you discovered in the previous page.

THEOREM 4.2 SUMMATION FORMULAS

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example Evaluate the following sum for $n = 10, 100, 1000, and\ 10000$.

$$\sum_{i=1}^{n} \frac{i+1}{n^2}$$

Back to Area…

Okay, so what does this have to do with area? When we are finding an approximation for the are under a curve, we are just doing repeated addition of many small areas – That's a summation!

Try it out! Approximate the area under the curve given by $f(x) = 17 - x^2$ between $x = 0$ and $x = 4$ using five rectangles. Find an Upper Sum and a Lower Sum.

Lower Sum:

Upper Sum: