Calculus

Name: Date:

3C-1: Concavity and the 2nd Derivative

We can now find critical points, increasing and decreasing intervals, and use these to find the minimums and maximums. We will now use the 2nd derivative to find the change in the first derivative and get a better picture of the graph of a function.

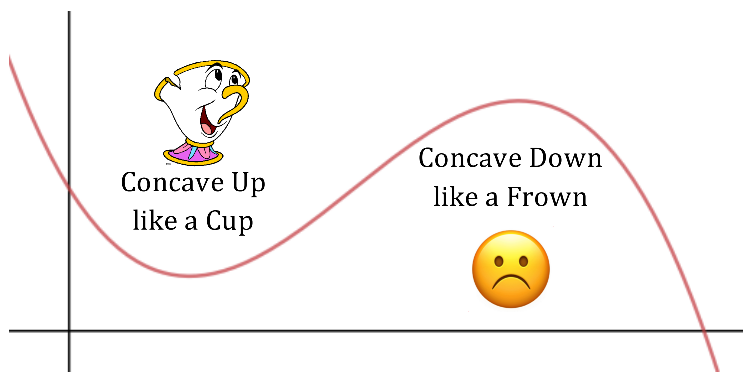
What does the 2nd Derivative Tell About the Graph?

Definition: Concavity

If f is differentiable on an open interval I, then the graph of f is

Concave upward on I if f' is increasing on the interval, and

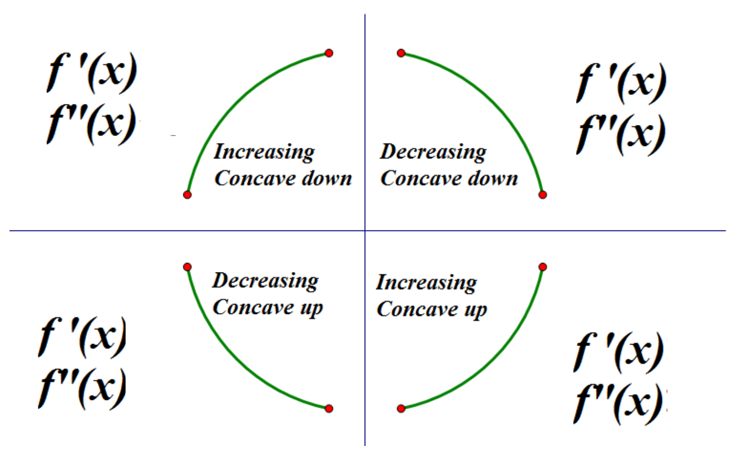
Concave downward on I if f' is decreasing on the interval.



So, concavity is a result of the change in the slope of the graph, also known as the rate of change in the derivative. When we are finding the rate of change in the derivative, this is the "derivative of the derivative"… which is the 2nd derivative.

A picture to remember:

Consider the first and second derivative for each of these curves.



When a function is **concave up** on an interval,

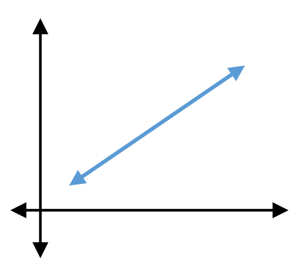
$f''(x)$ is ______________

When a function is **concave down** on an interval,

$f''(x)$ is ______________

When a function is **linear** on an interval,

$f''(x)$ is ______________



Want to find concavity?

Use the second derivative!

Make another number line!

Try it out: Consider the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x$.

a. Use the 1st derivative to find the critical points:

b. Use the 2nd derivative to determine its concavity:

c. Sketch a rough graph of $f(x)$

Notice that something happens to the concavity at $x = 1$. This is a point where it changes from concave down to concave up. This is called a **point of inflection** where the concavity changes.

Inflection points:

If f is a continuous function on an open interval that includes c, then the point $(c, f(c))$ if the concavity changes from upward to downward (or downward to upward) at this point.

- A Point of Inflection (p.i.v.) can occur when $f''(c) = 0$ or $f''(c)$=DNE.
  However, these conditions do not guarantee an inflection point.
- **Important note:** A function can change concavity at a discontinuity, however, this point *is not* an inflection point because the function is undefined at this value

Test for Concavity:

1. Find any discontinuities $f(x)$
2. Identify any PIVs by finding where $f''(x) = 0$ or $f''(x) = DNE$.
3. Label all these values on a number line and find the sign of $f''(x)$
   If $f''(x) > 0$, then f is concave upward on the interval.
   If $f''(x) < 0$, then f is concave downward on the interval.

Okay, let's try some out.

Try it! Find the PIVs and the open intervals of concavity.

a. $y = 3 + \sin x, \ \ x \in [0, 2\pi]$

b. $f(x) = 6(x^2 + 3)^{-1}$

c. $g(x) = \frac{x^2+1}{x^2-1}$

d. $f(x) = x^4 - 4x^3$

e. $y = e^{-x^2}$

f. $g(x) = x^4$

Putting it all together now, we know:

1. If $f'(c) = 0$, then there is a minimum or a maximum at $x = c$, and
2. The second derivative $f"(c)$ tells us if it's concave up or down.

So, how does this gives us the second derivative test (which is nice, but not always necessary because the first derivative test usually tells us enough info):

THEOREM 3.9 SECOND DERIVATIVE TEST

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.